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INVESTIGATION OF THE QUALITATIVE BEHAVIOR OF THE EQUILIBRIUM POINTS FOR A MODIFIED LOTKA-VOLTERRA MODEL

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ABSTRACT

We are interested in a modified Lotka-Volterra model to analyze population dynamics of two competing species which are ecologically identical (that is, they use the same resource). The model incorporates a non-linear relationship to represent the interaction between the species. We study the stability of the equilibrium points of the system and compare the qualitative behavior of the equilibrium points in our model with qualitative behavior of the classical Lotka-Volterra equations. Our result suggests that in some cases the modified model may have more than one equilibrium points in the interior of the first quadrant, which biologically means that the two species may co-exist at multiple positive population sizes.

INTRODUCTION

One of the first mathematical models to incorporate interactions between predators and prey was proposed in 1925 by the American biophysicist Alfred Lotka and the Italian mathematician Vito Volterra. The model is one of the earliest predator-prey models to be based on sound mathematical principles. It can be used to model competition between two species which are ecologically identical (use the same resource) and to describe the possible effects of the competition in terms of coexistence or competitive exclusion [4]. It forms the basis of many models used today in the analysis of population dynamics; the model has also been applied to various problems in population biology, chemical kinetics, neural networks and epidemiology, and has become a classic example for nonlinear dynamical systems.

Competition occurs when animals utilize common resources that are in short supply; otherwise, if resources are not in short supply, competition occurs when the animals seeking that resource harm one another in the process. There are three important points associated with this definition. First, the interaction between two species will be reciprocal, meaning that

it will cause demonstrable reductions in survival, growth, or fecundity of the species, although one species usually has a bigger affect overall. Secondly, a resource is in short supply. Even if animals overlap completely in resource utilization, competition usually does not occur unless a resource is limited in some way. Finally, as can be deduced from the above, the competition is density dependent. To model these phenomena, the Lotka-Volterra competition model incorporates logistic components to model intraspecific competition (competition among members of the same species) and other terms ($-\mu_1x$ and $-\mu_2y$ in equation (1.1)) to incorporate the effects of interspecific competition (competition among two or more species).

The following is the classical Lotka-Volterra model:

$$\begin{aligned}\frac{dx}{dt} &= \beta_1x(K_1 - x - \mu_1y) \\ \frac{dy}{dt} &= \beta_2y(K_2 - y - \mu_2x) \\ x(0) &\geq 0 \text{ and } y(0) \geq 0\end{aligned}\tag{1.1}$$

The two variables $x(t)$ and $y(t)$ represent the number of individuals (or population density) of species x and y at time t ; the β_i parameters are the intrinsic growth rates for the two species x and y ; the parameters K_i are the carrying capacities for the two species; the μ_i parameters are the coefficients of competition, which measure the competitive effect of one species on the other. As illustrated in the model, population y interferes with population x negatively in a linear fashion and vice versa. In the absence of one population, the other population grows based on logistic law. For example, if $y=0$ then equation (1.1) will be reduced to

$$\frac{dx}{dt} = \beta_1x(K_1 - x), \quad x(0) > 0,$$

which is a logistic population growth model. The term μ_1x (or μ_2y) can be thought of as the contribution made by population x (or y) to a decline in the growth rate of population y (or x).

The classical Lotka-Volterra competition model has been modified by a number of research studies. For example, Al-Omari and Gourley [1] modified the model to incorporate a time-delay between birth and maturity and assume that only adult members of each species compete, while Liu [2] studied a spatial stochastic version of the Lotka-Volterra equation. In [3], Taylor and Crizer proposed a modified version of the classical Lotka-Volterra competition model to represent the interaction between species by incorporating a non-linear relationship.

In [3], Taylor and Crizer showed that both the classical model and their modified model share the points $(0,0)$, $(K_1,0)$, and $(0,K_2)$ as equilibrium points and that the stability of these points for both models is the same. For some parameter values, they also showed that the modified model has a unique

fourth equilibrium point (x_c, y_c) in the first quadrant where the two populations co-exist.

In this paper, we studied the modified model proposed by Taylor and Crizer [3] and we proved the existence of multiple equilibrium points in the first quadrant at which the two populations co-exist. We studied also the stability of the equilibrium points.

The Classical Lotka-Volterra Competition Model

The equilibrium points of the classical Lotka-Volterra competition model (equation (1.1)) are $E_0=(0,0)$, $E_1=(K_1,0)$, $E_2=(0,K_2)$, as well as the solution to the following equation in the interior of the first quadrant:

$$\begin{cases} x + \mu_1 y = K_1 \\ y + \mu_2 x = K_2 \end{cases} \tag{1.2}$$

In Figure 1, four possible cases regarding the solution of equation (1.2) in the first quadrant are shown.

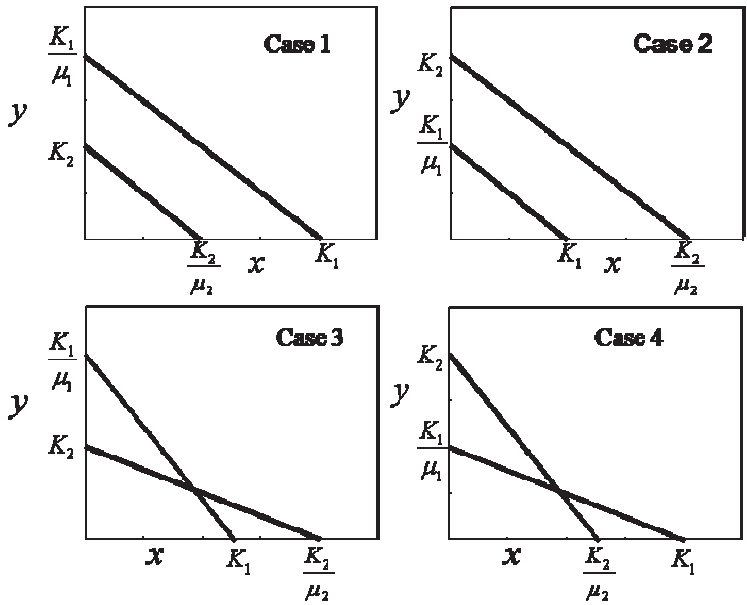


Figure 1: Four cases for the solution of equation (1.2) in the first quadrant.

The dynamics of the classical Lotka-Volterra model is well known [3]. The following are the main properties of the system in equation (1.1):

- (P1) The solutions are positive if the initial conditions are positive.
- (P2) The solutions are bounded.
- (P3) The system is monotonic.
- (P4) The system has at most four equilibria: the extinction equilibrium $E_0=(0,0)$, the exclusive equilibria $E_1=(K_1,0)$, and $E_2=(0,K_2)$, and possibly the coexistence equilibrium $E_3=((K_1-K_2\mu_1)/(1-\mu_1\mu_2), (K_2-K_1\mu_2)/(1-\mu_1\mu_2))$. The equilibrium E_3 exists only in case 3, where $\frac{K_1}{\mu_1} > K_2$ and $\frac{K_2}{\mu_2} > K_1$, or in case 4, where $\frac{K_1}{\mu_1} < K_2$ and $\frac{K_2}{\mu_2} < K_1$.
- (P5) The stability of the equilibria can be summarized as follows:

Case 1: If $\frac{K_1}{\mu_1} > K_2$ and $\frac{K_2}{\mu_2} < K_1$ then E_0 is a repeller and is always unstable, E_1 is saddle, and E_2 is a stable equilibrium.

Case 2: If $\frac{K_1}{\mu_1} < K_2$ and $\frac{K_2}{\mu_2} > K_1$ then E_0 is a repeller and is always unstable, E_1 is a stable equilibrium, and E_2 is saddle.

Case 3: If $\frac{K_1}{\mu_1} > K_2$ and $\frac{K_2}{\mu_2} < K_1$ then E_0 is a repeller and is always unstable, E_1 and E_2 are saddle and the interior equilibrium E_3 is a stable equilibrium.

Case 4: If $\frac{K_1}{\mu_1} < K_2$ and $\frac{K_2}{\mu_2} < K_1$ then E_0 is a repeller and is always unstable, E_1 and E_2 are stable equilibria and the interior equilibrium E_3 is saddle.

In case 3, E_3 is stable equilibrium, which biologically implies that the two populations will eventually co-exist at E_3 if the initial size of the two populations is on the basin of attraction of E_3 . In case 3 the basin of attraction of E_3 is the entire first quadrant except those points on x-axis and the y-axis. Every trajectory started in the first quadrant, except those starting with $x=0$ or $y=0$, will tend toward the stable equilibrium E_3 , that is,

$$\lim_{t \rightarrow \infty} x(t) = \frac{K_1 - K_2\mu_1}{1 - \mu_1\mu_2} \quad \text{and} \quad \lim_{t \rightarrow \infty} y(t) = \frac{K_2 - K_1\mu_2}{1 - \mu_1\mu_2}.$$

In case 4, both E_1 and E_2 are stable equilibria and E_3 is a saddle. Thus, if the initial size of the two populations is on the basin of attraction of E_1 or

E_2 then one population will survive and the other population will become extinct; and if the initial population is on the stable manifold of E_3 then both populations will co-exist at the saddle E_3 . The stable manifold of E_3 separates the basin of attraction of the stable equilibria E_1 and E_2 . The phase portraits for each case are shown in Figure 2 below.

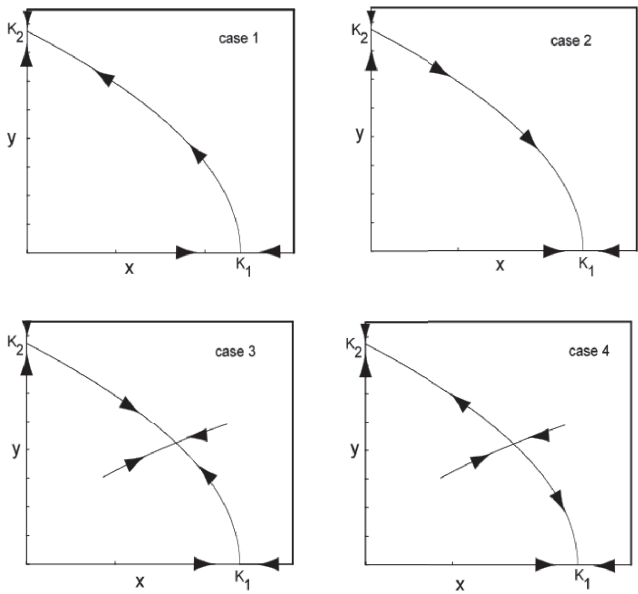


Figure 2: Phase portraits for classical Lotka-Volterra competition model.

Modified Model of the Lotka-Volterra Competition Model

The classical Lotka-Volterra model incorporates interspecific competition by using the linear terms $-\mu_1 y$ and $-\mu_2 x$. x' is negatively affected by the term $-\mu_1 y$; that is, population y affects the growth of population x negatively in a linear fashion and vice versa. It is more realistic to assume, however, that as one population grows it becomes more efficient than the other at gathering the shared resource. Competition models with nonlinear interactions and their biological significance have been studied by several authors [5]. Taylor and Crizer (2005) proposed a modified Lotka-Volterra model which incorporates a nonlinear relationship between the two species. They assume population y interferes with population x negatively in a quadratic fashion and vice versa. x' is negatively affected by the term $-\mu_1 y^2$ and y' is negatively affected by the term $-\mu_2 x^2$. If y is large, then x' , which is the rate of change x , is small. Thus, population y has a large influence on the growth of population x . On the other hand, if y is small then x' is large and, consequently, population y has smaller influence on the growth of population x .

The modified Lotka-Volterra competition model with nonlinear terms is given below:

$$\begin{aligned}\frac{dx}{dt} &= \beta_1 x(K_1 - x - \mu_1 y^2) \\ \frac{dy}{dt} &= \beta_2 y(K_2 - y - \mu_2 x^2) \\ x(0) &\geq 0 \text{ and } y(0) \geq 0\end{aligned}\tag{1.3}$$

where again $\beta_i, K_i, \mu_i, i=1,2$, are positive constants and have the same definition as in the classical model.

In their paper, Taylor and Crizer [3] assumed that the classical model and the modified model defined by equation (1.3) have at most four equilibrium points. For both models, they studied the stability of the equilibrium points and they discussed conditions that determine the stability of the equilibrium point E_3 in the interior of the first quadrant.

In this paper we showed that, in some cases, the system (1.3) can have more than one equilibrium point in the interior of the first quadrant and we studied the stability of the equilibria.

Model Analysis

To analyze the modified model, the following results, which are proved in [3], will be useful.

Lemma 1: Equation (1.3) satisfies the following properties:

- (a) Every trajectory that starts in the first quadrant stay in the first quadrant.
- (b) Solutions in the first quadrant are bounded.
- (c) There is no periodic orbit in the first quadrant.

$E_0=(0,0)$, $E_1=(K_1,0)$, and $E_2=(0,K_2)$ are equilibrium points of equation (1.3) as well as the possibility of additional equilibrium point(s) which is/are positive solution(s) to the following equation:

$$\begin{aligned}x + \mu_1 y^2 &= K_1 \\ y + \mu_2 x^2 &= K_2\end{aligned}\tag{1.4}$$

Equation (1.4) is quadratic in both x and y and thus can possibly have up to four different solutions in the first quadrant. Thus, the equilibrium(s) (x_c, y_c) , which is(are) in the interior of the first quadrant, satisfy(s) the following fourth degree polynomials which be derived by using equation (1.4).

$$\mu_1 \mu_2^2 x_c^4 - 2K_2 \mu_1 \mu_2 x_c^2 + x_c + \mu_1 K_2^2 - K_1 = 0 \quad (1.5)$$

$$\mu_1^2 \mu_2 y_c^4 - 2K_1 \mu_1 \mu_2 y_c^2 + y_c + \mu_2 K_1^2 - K_2 = 0 \quad (1.6)$$

Using a combination of Descartes' rule of signs and various geometric arguments for the roots of the equations, we proved the following theorem:

Theorem 1: (1) If $\sqrt{\frac{K_2}{\mu_2}} < K_1$ and $\sqrt{\frac{K_1}{\mu_1}} > K_2$ then equation (1.4) has either no solution in the first quadrant or has exactly two solutions in the first quadrant.
 (2) If $\sqrt{\frac{K_2}{\mu_2}} > K_1$ and $\sqrt{\frac{K_1}{\mu_1}} < K_2$ then equation (1.4) has either no solution in the first quadrant or has exactly two solutions in the first quadrant.
 (3) If $\sqrt{\frac{K_2}{\mu_2}} > K_1$ and $\sqrt{\frac{K_1}{\mu_1}} > K_2$ then equation (1.4) has either exactly one solution in the first quadrant or has exactly three solutions in the first quadrant.
 (4) If $\sqrt{\frac{K_2}{\mu_2}} < K_1$ and $\sqrt{\frac{K_1}{\mu_1}} < K_2$ then equation (1.4) has exactly one solution in the first quadrant.

Proof: (1) From $\sqrt{\frac{K_2}{\mu_2}} < K_1$ and $\sqrt{\frac{K_1}{\mu_1}} > K_2$ we have that $\mu_2 K_1^2 - K_2 > 0$ and $\mu_1 K_2^2 - K_1 < 0$. In the polynomial $p_1(x) = \mu_1 \mu_2^2 x^4 - 2K_2 \mu_1 \mu_2 x^2 + x + \mu_1 K_2^2 - K_1$, (which is defined by the left hand side of equation 1.5) there are three sign changes among the coefficients $\mu_1 \mu_2^2$, $-2K_2 \mu_1 \mu_2$, 1, and $(\mu_1 K_2^2 - K_1)$; therefore, by Descartes' rule of signs, there are either three positive zeroes or there is one positive zero for the polynomial. Similarly, for the polynomial $p_2(y) = \mu_1^2 \mu_2 y^4 - 2K_1 \mu_1 \mu_2 y^2 + y + \mu_2 K_1^2 - K_2$, (which is defined by the left hand side of equation 1.6) there are two sign changes, and therefore either two positive zeroes or no positive zeros for the polynomial. Therefore, the two curves of the two polynomials intersect at most two times in the interior of the first quadrant. The curves defined by $x + \mu_1 y^2 = K_1$ and by $x + \mu_2 y^2 = K_2$ are monotone decreasing in the first quadrant ($\frac{dy}{dx} < 0$). Thus, the two equations and by define as a one to one function of . Thus, from the location of the

x-intercept and y-intercepts of the two curves, we can conclude that the two curves can't intersect exactly once in the first quadrant. Therefore, equation (1.4) has either no solution in the interior of first quadrant or has exactly two solutions in the interior of first quadrant.

(2) From $\sqrt{\frac{K_2}{\mu_2}} > K_1$ and $\sqrt{\frac{K_1}{\mu_1}} < K_2$, we have $\mu_2 K_1^2 - K_2 < 0$ and $\mu_1 K_2^2 - K_1 > 0$. Using Descartes' rule of signs for the polynomials defined in the proof of (1), by similar argument as in the proof of (1), equation (1.4) has either no solution in the interior of first quadrant or has exactly two solutions in the interior of first quadrant.

(3) Since $\sqrt{\frac{K_2}{\mu_2}} < K_1$ and $\sqrt{\frac{K_1}{\mu_1}} < K_2$ we have $\mu_2 K_1^2 - K_2 > 0$ and $\mu_1 K_2^2 - K_1 > 0$. Using Descartes' rule of signs for the polynomials defined in the proof of (1), equation (1.4) can have at most three positive solutions in the first quadrant. The curves defined by $x + \mu_1 y^2 = K_1$ and by $x + \mu_2 y^2 = K_2$ are monotone decreasing in the first quadrant. Thus, the two equations $x + \mu_1 y^2 = K_1$ and by $x + \mu_2 y^2 = K_2$ define y as a one to one function of x . From the location of the x-intercepts and y-intercepts of the two curves we can conclude that the two curves can't intersect exactly two times in the first quadrant and they must intersect. Hence equation (1.4) has either exactly one solution in the first quadrant or has exactly three solutions in the first quadrant.

(4) Since $\sqrt{\frac{K_2}{\mu_2}} < K_1$ and $\sqrt{\frac{K_1}{\mu_1}} < K_2$, we have $\mu_2 K_1^2 - K_2 > 0$ and $\mu_1 K_2^2 - K_1 > 0$. Using Descartes' rule of signs for the polynomials defined in the proof of (1), equation (1.4) can have at most two positive solutions in the first quadrant. The curves defined by $x + \mu_1 y^2 = K_1$ and by $x + \mu_2 y^2 = K_2$ are monotone decreasing in the first quadrant. Thus, the two equations $x + \mu_1 y^2 = K_1$ and by $x + \mu_2 y^2 = K_2$ define y as a one to one function of x . From the location of the x-intercepts and y-intercepts of the two curves we can conclude that the two curves can't intersect exactly two times in the first quadrant and they must intersect. Hence equation (1.4) has exactly one solution in the first quadrant.

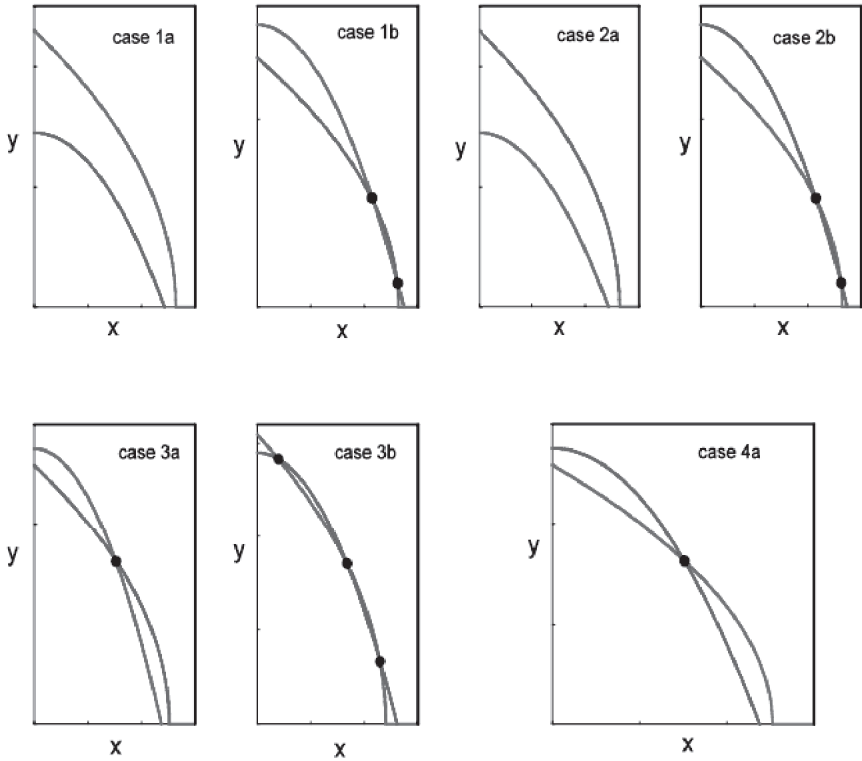


Figure 3: Four cases based on the curves defined by $x+\mu_1y^2=K_1$ and $x+\mu_2y^2=K_2$. The dots are intersections of the two curves.

Figure 3 shows four cases for the number possible intersections in the interior of the first quadrant for the curves defined by $x+\mu_1y^2=K_1$ and $x+\mu_2y^2=K_2$. Numerical examples of all possible scenarios for all cases are given below:

Case 1: (a) If $K_1=1.32$, $K_2=1.45$, $\mu_1=0.25$, and $\mu_2=0.98$ then equation (1.4) has no solution in the interior of the first quadrant.

(b) If $K_1=1.32$, $K_2=1.5$, $\mu_1=0.55$, and $\mu_2=0.98$ then equation (1.4) has exactly two solutions in the interior of the first quadrant:

$$E_1 = (0.60787, 1.1379) \text{ and } E_2 = (0.09797, 1.4906).$$

Case 2: (a) If $K_1=1.32$, $K_2=1.65$, $\mu_1=0.75$, and $\mu_2=0.8$ then equation (1.4) has no solution in the interior of the first quadrant.

(b) If $K_1=1.32$, $K_2=1.5$, $\mu_1=0.75$, and $\mu_2=0.8$ then equation (1.4) has exactly two solutions in the interior of the first quadrant:

$$E_1 = (1.19518, 1.4176), E_2 = (1.14111, 0.34318) \text{ and } E_3 = (0.84686, 0.84041).$$

Case 3: (a) If $K_1=1.15$, $K_2=1.45$, $\mu_1=0.5$, and $\mu_2=0.85$ then equation (1.4) has exactly one solution in the interior of the first quadrant: $E_1 = (0.11498, 1.4388)$

(b) If $K_1=1.2$, $K_2=1.45$, $\mu_1=0.5$, and $\mu_2=0.85$ then equation (1.4) has exactly three solutions in the first quadrant:

$$E_1 = (0.19518, 1.4176) \text{ and } E_2 = (0.84686, 0.84041).$$

Case 4: If $K_1=1.26$, $K_2=1.38$, $\mu_1=0.75$, and $\mu_2=0.98$ then equation (1.4) has exactly one solution in the interior of the first quadrant: $E_1 = (0.76521, 0.81959)$

Stability Analysis

The stability of the equilibrium points $E_0=(0,0)$, $E_1=(K_1,0)$, and $E_2=(0,K_2)$ of equation (1.3) is the same as the classical model for all cases [3]. For the equilibrium points $E=(x_c, y_c)$ in the interior of the first quadrant, as shown in [3], the Jacobian matrix is:

$$\begin{pmatrix} -\beta_1 x_c & -2\beta_1 \mu_1 x_c y_c \\ -2\beta_2 \mu_2 x_c y_c & -\beta_2 y_c \end{pmatrix} \quad (1.7)$$

The characteristic polynomial of the matrix is:

$$p(\lambda) = \lambda^2 + (\beta_1 x_c + \beta_2 y_c) \lambda + \beta_1 \beta_2 x_c y_c (1 - 4\mu_1 \mu_2 x_c y_c) \quad (1.8)$$

The roots of the polynomial are the eigenvalues of the Jacobian matrix, and they are:

$$\lambda_{1,2} = \frac{-(\beta_1 x_c + \beta_2 y_c) \pm \sqrt{(\beta_1 x_c + \beta_2 y_c)^2 + 16\beta_1 \beta_2 \mu_1 \mu_2 x_c^2 y_c^2}}{2} \quad (1.9)$$

The two eigenvalues are real. The stability of the equilibrium(s) $E=(x_c, y_c)$ depends on the sign of the two eigenvalues. If both eigenvalues are positive then $E=(x_c, y_c)$ is unstable; if both eigenvalues are negative then $E=(x_c, y_c)$ is stable; and if the eigenvalues have opposite signs then $E=(x_c, y_c)$ is saddle. Using Descartes' rule of signs for the roots of equation (1.8), we can conclude that the sign of $1-4\mu_1\mu_2 x_c y_c$ determines the sign of the eigenvalues. The sign of $1-4\mu_1\mu_2 x_c y_c$ has no direct relationship with the parameters K_1 , K_2 , μ_1 and μ_2 . However, the stability of the equilibrium points can be determined using qualitative analysis. Figure 4 illustrates the phase portraits of the modified model for all cases.

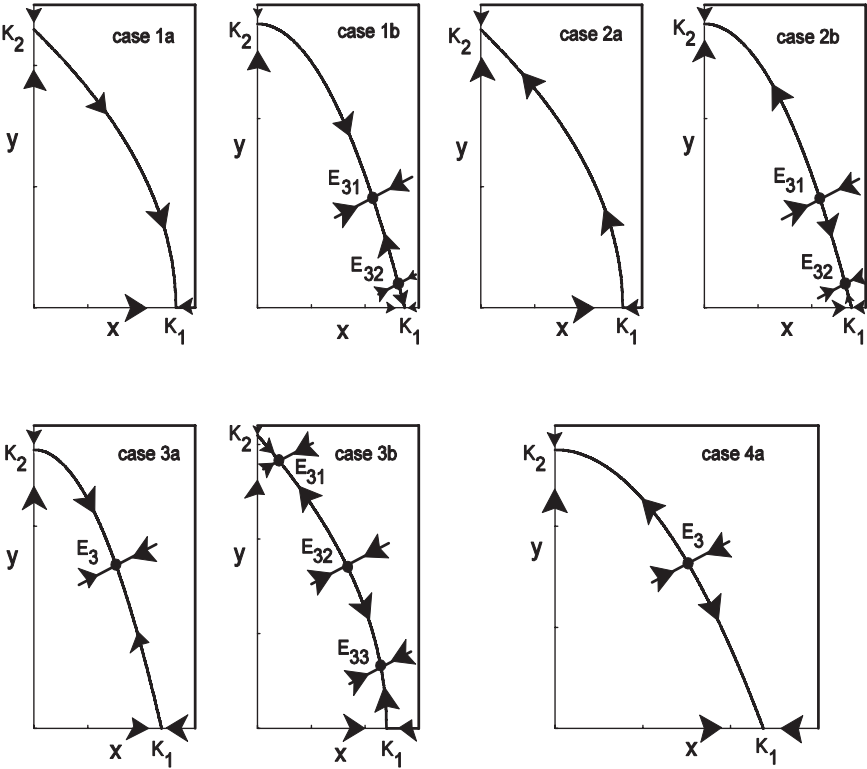


Figure 4: Phase portraits for the modified Lotka-Volterra competition model.

The following are summary of the stability of the equilibrium points of the modified Lotka-Volterra competition model:

Case 1: $\sqrt{\frac{K_2}{\mu_2}} < K_1$ and $\sqrt{\frac{K_1}{\mu_1}} > K_2$:

In this case, regardless of the number of equilibrium points in the interior first quadrant, the stability of the equilibrium points $E_0=(0,0)$, $E_1=(K_1,0)$, and $E_2=(0,K_2)$ is the same as their stability in the classical model. $E_0=(0,0)$ is always an unstable equilibrium (repeller), $E_1=(K_1,0)$ is a stable equilibrium and $E_2=(0,K_2)$ is a saddle point. As shown in Figure 4 the system (1.3) either has no equilibrium point in the interior of first quadrant (Case 1a, which is the same as the case in the classical model) or it has two equilibrium points in the interior first quadrant (Case 1b). In Case1b, the upper interior equilibrium E_{31} is stable and the lower interior equilibrium E_{32} is a saddle. Thus, if the initial population size is on the basin of attraction of E_{31} then the two populations eventually will co-exist at E_{31} ; otherwise, if the initial population

size is on the stable manifold of E_{32} then the two populations eventually will co-exist at E_{32} . The stable manifold of E_{32} separates the basin of attraction of the stable equilibria E_{31} and E_1 .

Case 2: $\sqrt{\frac{K_2}{\mu_2}} > K_1$ and $\sqrt{\frac{K_1}{\mu_1}} < K_2$:

As shown in Figure 4, in this case, the system (1.3) may have either no equilibrium point in the interior first quadrant (Case 2a, which is the same as the case in the classical model) or it may have two such equilibrium points (Case 2b). Regardless of the number of equilibria in the interior first quadrant, $E_0=(0,0)$ is always an unstable equilibrium, $E_2=(0,K_2)$ is a stable equilibrium and $E_1=(K_1,0)$ is a saddle point. In Case 2b, the upper equilibrium point in the interior first quadrant E_{31} is saddle and the lower interior equilibrium E_{32} is a stable equilibrium. Biologically it means that if the initial population size of both x and y is on the stable manifold of E_{31} then the two populations eventually will co-exist at E_{31} . Similarly, if the initial population size of both x and y is on the basin of attraction of the stable equilibrium point E_{32} then the two populations eventually will co-exist at E_{32} .

Case 3: $\sqrt{\frac{K_2}{\mu_2}} > K_1$ and $\sqrt{\frac{K_1}{\mu_1}} < K_2$:

In this case the system (1.3) may have either exactly one equilibrium point in the interior first quadrant (Case 3a) or exactly three equilibria in the interior first quadrant (Case 3b), both of which can be seen in Figure 4. In case 3a and case 3b, the boundary equilibria $E_0=(0,0)$, $E_1=(K_1,0)$, and $E_2=(0,K_2)$ have the same stability as in the classical model. $E_0=(0,0)$ is always an unstable equilibrium and $E_2=(0,K_2)$ and $E_1=(K_1,0)$ are both saddle points. In case 3a (which is the same as the case in the classical model), the interior equilibrium E_2 is stable. In case 3b, the interior equilibrium points E_{31} and E_{33} are both stable and the middle equilibrium E_{32} is saddle. The stable manifold of E_{32} separates the basin of attractions of the two stable equilibria E_{31} and E_{33} ; thus, depending the parameter values, if the initial population size is on the interior of the first quadrant then the two population may co-exist at either E_{31} , E_{32} or E_{33} .

Case 4: $\sqrt{\frac{K_2}{\mu_2}} < K_1$ and $\sqrt{\frac{K_1}{\mu_1}} < K_2$:

In this case there is only one interior equilibrium point E_3 . As shown in Figure 4, the stability of the boundary equilibria $E_0=(0,0)$, $E_1=(K_1,0)$, and $E_2=(0,K_2)$ and the stability of E_3 is the same as in the classical model.

Using Matlab, the trajectories for specific parameter values were graphed and are given in Figure 5 and Figure 6. These graphs match the analysis given above. In Figure 5 the parameter values satisfy the inequalities $\frac{K_1}{\mu_1} > K_2$ and

$\frac{K_2}{\mu_2} < K_1$ and $\sqrt{\frac{K_2}{\mu_2}} > K_1$ and $\sqrt{\frac{K_1}{\mu_1}} > K_2$; for these parameter values, the classical model predicts the dynamics of case 3 and the modified model predicts the dynamics in case 2b. Similarly, in Figure 6, the parameter values satisfy the inequalities $\frac{K_1}{\mu_1} > K_2$ and $\frac{K_2}{\mu_2} > K_1$ and $\sqrt{\frac{K_2}{\mu_2}} > K_1$ and $\sqrt{\frac{K_1}{\mu_1}} > K_2$; the classical model predicts case 3 and the modified model predicts case 3b. In both figures the two models predict different population dynamics.

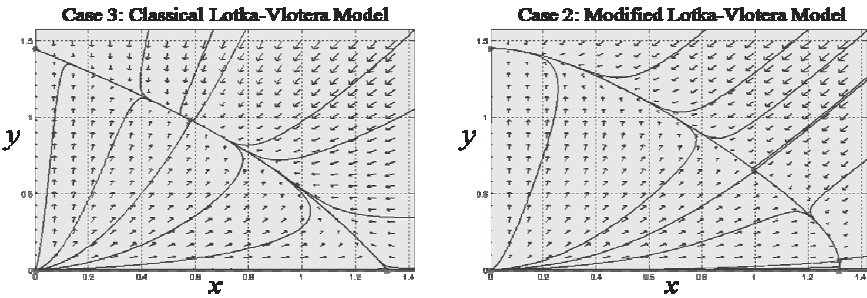
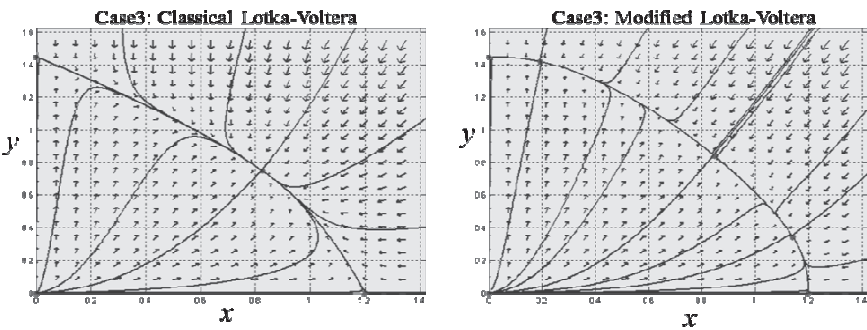


Figure 5: Phase portraits, equilibrium points (the red dots), and sample trajectories created by Matlab 7.0.1 software package for parameter values $\beta_1 = 0.2$, $\beta_2 = 0.3$, $K_1 = 1.32$, $K_2 = 1.454$, $\mu_1 = 0.75$, and $\mu_2 = 0.8$.

These parameter groups in Figure 5 & 6 satisfy the inequalities in Case 3 for the Classical model and Case 2 for the Modified model.

Figure 6: Phase portraits, equilibrium points (the red dots), and sample trajectories created by Matlab 7.0.1 software package for parameter values and These parameter groups satisfy the inequalities in Case 3 for both the Classical and the Modified models.



Conclusion

In the modified Lotka-Volterra model, we showed that, for some parameter values, the model predicts the existence of multiple equilibrium points in the interior of the first quadrant at which the two populations co-exist (depending on the initial size of the two populations). In the classical model, there exists at most one equilibrium point in the interior of the first quadrant. As shown in Figures 5 and Figure 6, depending on the parameter values, the two models predict different dynamics.

In [3], Taylor and Crizer showed that both the classical model and the modified model share $(0,0)$, $(K_1,0)$, and $(0,K_2)$ as equilibrium points and showed that the stability of these points for both models is the same; they assume that in case 2 and case 3, there exists a unique fourth equilibrium point (x_c, y_c) in the first quadrant where the two populations co-exist. In this paper we discussed the existence of multiple equilibrium points in the interior of the first quadrant under certain conditions. Consequently, depending on the values of the parameters K_1 , K_2 , μ_1 , and μ_2 , our analysis reveals more complex dynamics in cases 1, 2 and 3. In case 4, the stability and number of equilibria for both models are the same.

In the classical model, the stability of the equilibrium in the interior of the first quadrant depends on the expression $1 - \mu_1 \mu_2$; in the modified model, however, the stability of the equilibrium point(s) in the interior of the first quadrant depends on the sign of the expression $1 - 4 \mu_1 \mu_2 x_c y_c$. The sign of $1 - 4 \mu_1 \mu_2 x_c y_c$ cannot be determined directly from the inequalities in the four cases involving the parameters K_1 's and μ_1 's; instead, we use geometric qualitative analysis to determine the stability of these equilibrium points.

The modified model is based on more realistic assumptions that, as one population grows, it becomes more efficient at gathering the resource shared with the other population. The modified model possesses richer dynamics than does the classical model and thus can potentially be more useful in describing the interaction between competing species. The main conclusion in this paper is that for some parameter values the modified model has multiple positive equilibria. The next immediate research step is to obtain complete analytic formulas in terms of the model parameters for the equilibria, and to study the stability of the equilibria through bifurcation analysis.

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